Specialist Mathematics Unit 1: **Chapter 6**

EX 6A

1. (a) $\mathbf{r}_A = 2\mathbf{i} + 3\mathbf{j}$
   (b) $\mathbf{r}_B = 4\mathbf{i} + 5\mathbf{j}$
   (c) $\mathbf{r}_C = \mathbf{i} + 4\mathbf{j}$
   (d) ${}_A\mathbf{r}_B = \mathbf{r}_A - \mathbf{r}_B = -2\mathbf{i} - 2\mathbf{j}$
   (e) ${}_A\mathbf{r}_C = \mathbf{r}_A - \mathbf{r}_C = \mathbf{i} - \mathbf{j}$
   (f) ${}_A\mathbf{r}_D = \mathbf{r}_A - \mathbf{r}_D = -2\mathbf{i} + 2\mathbf{j}$
   (g) ${}_D\mathbf{r}_A = \mathbf{r}_D - \mathbf{r}_A = 2\mathbf{i} - 2\mathbf{j}$
   (h) ${}_B\mathbf{r}_C = \mathbf{r}_B - \mathbf{r}_C = 3\mathbf{i} + \mathbf{j}$
   (i) ${}_C\mathbf{r}_D = \mathbf{r}_C - \mathbf{r}_D = -3\mathbf{i} + 3\mathbf{j}$
   (j) ${}_D\mathbf{r}_C = \mathbf{r}_D - \mathbf{r}_C = 3\mathbf{i} - 3\mathbf{j}$

2. (a) $\mathbf{r}_A$ is given
   (b) $\mathbf{r}_B$ is given
   (c) ${}_C\mathbf{r}_A = \mathbf{r}_C - \mathbf{r}_A$ so
   $\mathbf{r}_C = {}_C\mathbf{r}_A + \mathbf{r}_A = 2\mathbf{i} + \mathbf{j}$
   (d) $\mathbf{r}_D = {}_D\mathbf{r}_A + \mathbf{r}_A = \mathbf{i} + 3\mathbf{j}$
   (e) $\mathbf{r}_E$ is given
   (f) $\mathbf{r}_F = {}_F\mathbf{r}_E + \mathbf{r}_E = -\mathbf{i} + 5\mathbf{j}$
   (g) $\mathbf{r}_G = {}_G\mathbf{r}_D + \mathbf{r}_D = -3\mathbf{i} + 2\mathbf{j}$
   (h) $\mathbf{r}_H = {}_H\mathbf{r}_G + \mathbf{r}_G = -2\mathbf{i} + \mathbf{j}$
   (i) $\mathbf{r}_I = {}_I\mathbf{r}_A + \mathbf{r}_A = 5\mathbf{i} + 5\mathbf{j}$
   (j) ${}_I\mathbf{r}_J = \mathbf{r}_I - \mathbf{r}_J$ so
   $\mathbf{r}_J = \mathbf{r}_I - {}_I\mathbf{r}_J = 5\mathbf{i} + 2\mathbf{j}$
   (k) $\mathbf{r}_K = \mathbf{r}_H - {}_H\mathbf{r}_K = 3\mathbf{i}$
   (l) $\mathbf{r}_L = \mathbf{r}_I - {}_I\mathbf{r}_L = 4\mathbf{j}$

3. $\mathbf{r}_A = 7\mathbf{i} + 11\mathbf{j}$
   ${}_B\mathbf{r}_A = -12\mathbf{i} - 8\mathbf{j}$
   $\mathbf{r}_B = \mathbf{r}_A + {}_B\mathbf{r}_A = (-5\mathbf{i} + 3\mathbf{j})$km.

4. $\overrightarrow{AB}$ represents the displacement *from* A *to* B, thus it is the displacement of B relative to A.
   Note the importance of understanding this result:
   $$\overrightarrow{AB} = {}_B\mathbf{r}_A$$
   Some of you will get this the wrong way around if you are not careful!

5. $\mathbf{r}_A = 11\mathbf{i} - 3\mathbf{j}$
   ${}_A\mathbf{r}_B = -8\mathbf{i} - 8\mathbf{j}$
   $\mathbf{r}_B = \mathbf{r}_A - {}_A\mathbf{r}_B = 19\mathbf{i} + 5\mathbf{j}$
   $|\mathbf{r}_B| = \sqrt{19^2 + 5^2} = \sqrt{386}$km.

6. $\mathbf{r}_A = 5\mathbf{i} + 2\mathbf{j}$
   ${}_A\mathbf{r}_B = 8\mathbf{i} + 3\mathbf{j}$
   $\mathbf{r}_B = \mathbf{r}_A - {}_A\mathbf{r}_B = -3\mathbf{i} - \mathbf{j}$
   $|\mathbf{r}_B| = \sqrt{3^2 + 1^2} = \sqrt{10}$km.

7. $\mathbf{r}_A = 2\mathbf{i} + 4\mathbf{j}$
   ${}_C\mathbf{r}_A = -4\mathbf{i} - \mathbf{j}$
   $\mathbf{r}_C = \mathbf{r}_A + {}_C\mathbf{r}_A = -2\mathbf{i} + 3\mathbf{j}$
   $\mathbf{r}_B = \mathbf{r}_C + {}_B\mathbf{r}_C = 5\mathbf{i} - \mathbf{j}$

8. LHS:
$$\begin{aligned}
{}_D\mathbf{r}_E + {}_E\mathbf{r}_F &= (\mathbf{r}_D - \mathbf{r}_E) + (\mathbf{r}_E - \mathbf{r}_F) \\
&= \mathbf{r}_D - \mathbf{r}_E + \mathbf{r}_E - \mathbf{r}_F \\
&= \mathbf{r}_D - \mathbf{r}_F \\
&= {}_D\mathbf{r}_F \\
&= \text{RHS. Q.E.D.}
\end{aligned}$$

EX 6B

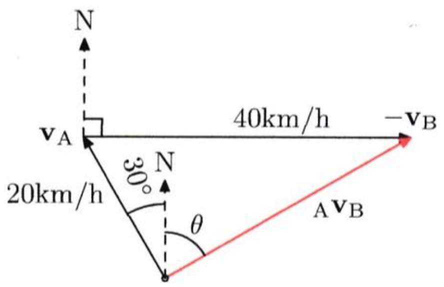
1. ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B = (2\mathbf{i} - 3\mathbf{j}) - (-4\mathbf{i} + 7\mathbf{j}) = 6\mathbf{i} - 10\mathbf{j}$

2. ${}_A\mathbf{v}_B = (4\mathbf{i} + 2\mathbf{j}) - (7\mathbf{i} - \mathbf{j}) = -3\mathbf{i} + 3\mathbf{j}$

3. ${}_A\mathbf{v}_B = (3\mathbf{i} - 2\mathbf{j}) - (4\mathbf{i} + 7\mathbf{j}) = -\mathbf{i} - 9\mathbf{j}$

4. ${}_A\mathbf{v}_B = (6\mathbf{i} + 2\mathbf{j}) - (3\mathbf{i} - \mathbf{j}) = 3\mathbf{i} + 3\mathbf{j}$

5. ${}_A\mathbf{v}_B = \mathbf{v}_A + (-\mathbf{v}_B)$



By the cosine rule,

$$\begin{aligned}
|{}_A\mathbf{v}_B|^2 &= 20^2 + 40^2 - 2 \times 20 \times 40 \cos 60° \\
|{}_A\mathbf{v}_B| &= 20\sqrt{3} \\
&\approx 36.64\text{km/h}
\end{aligned}$$

By the sine rule,

$$\frac{\sin(\theta + 30^\circ)}{40} = \frac{\sin 60^\circ}{20\sqrt{3}}$$

$$\sin(\theta + 30^\circ) = \frac{40 \times \frac{\sqrt{3}}{2}}{20\sqrt{3}}$$

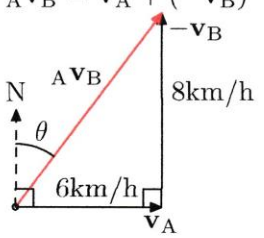
$$= 1$$

$$\therefore \quad \theta + 30^\circ = 90^\circ$$

$$\theta = 60^\circ$$

${}_A\mathbf{v}_B = 36.64$km/h on a bearing of 060°.

6. ${}_A\mathbf{v}_B = \mathbf{v}_A + (-\mathbf{v}_B)$



By Pythagoras,

$$|{}_A\mathbf{v}_B|^2 = 6^2 + 8^2$$

$$|{}_A\mathbf{v}_B| = 10\text{km/h}$$

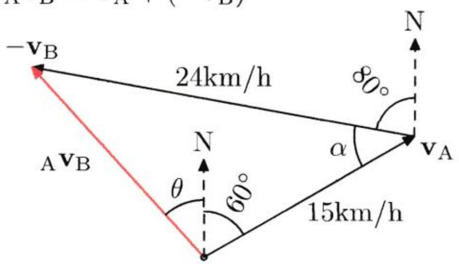
$$\tan(90^\circ - \theta) = \frac{8}{6}$$

$$90^\circ - \theta \approx 53.13^\circ$$

$$\theta \approx 36.87^\circ$$

${}_A\mathbf{v}_B = 10$km/h on a bearing of 037°.

7. ${}_A\mathbf{v}_B = \mathbf{v}_A + (-\mathbf{v}_B)$



$$60 + (80 + \alpha) = 180 \text{ (cointerior angles)}$$

$$\alpha = 40^\circ$$

By the cosine rule,

$$|{}_A\mathbf{v}_B|^2 = 24^2 + 15^2 - 2 \times 24 \times 15 \cos 40^\circ$$

$$|{}_A\mathbf{v}_B| \approx 15.79\text{km/h}$$

By the cosine rule again (since the sine rule would be ambiguous here),

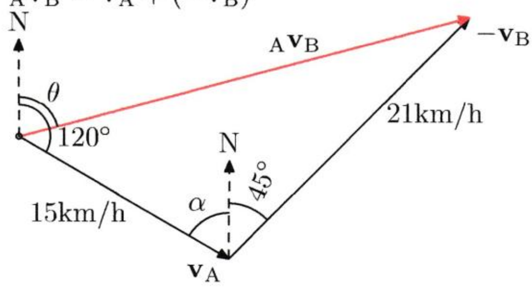
$$\theta + 60 = \cos^{-1} \frac{15^2 + 15.79^2 - 24^2}{2 \times 15 \times 15.79}$$

$$= 102.38^\circ$$

$$\theta = 42.38^\circ$$

${}_A\mathbf{v}_B = 15.8$km/h on a bearing of 318°.

8. ${}_A\mathbf{v}_B = \mathbf{v}_A + (-\mathbf{v}_B)$



$$\alpha = 180 - 120 \text{ (cointerior angles)}$$

$$= 60^\circ$$

$$\alpha + 45 = 105^\circ$$

By the cosine rule,

$$|{}_A\mathbf{v}_B|^2 = 21^2 + 15^2 - 2 \times 21 \times 15 \cos 105^\circ$$

$$|{}_A\mathbf{v}_B| = 28.79\text{km/h}$$

By the sine rule (unambiguous here),

$$\frac{\sin(120^\circ - \theta)}{21} = \frac{\sin 105^\circ}{28.79}$$

$$120^\circ - \theta = \sin^{-1} \frac{21 \sin 105^\circ}{28.79}$$

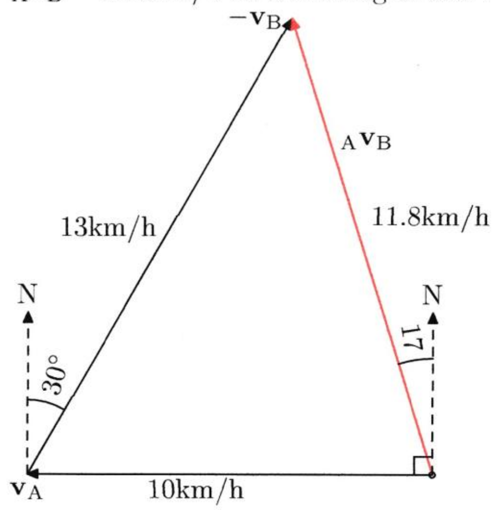
$$= 44.79^\circ$$

$$\theta = 120 - 44.79$$

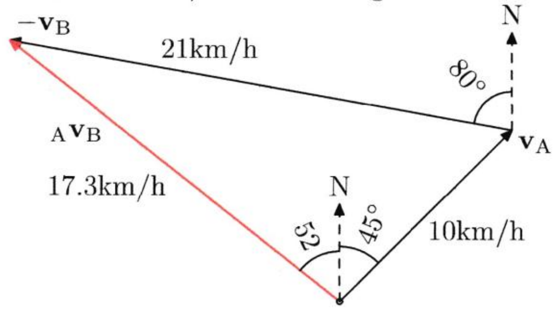
$$= 75.21^\circ$$

${}_A\mathbf{v}_B = 28.8$km/h on a bearing of 075°.

9. ${}_A\mathbf{v}_B = 11.8$km/h on a bearing of 343°.



10. ${}_A\mathbf{v}_B = 17.3$km/h on a bearing of 308°.

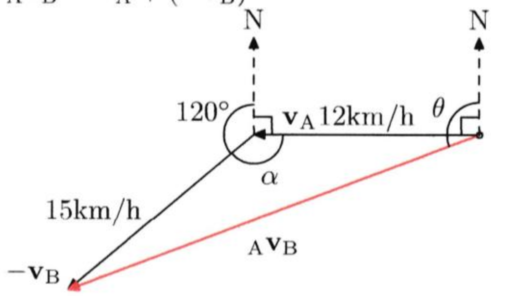


11. $\mathbf{v}_A = 7\mathbf{i} - 10\mathbf{j}$
$\mathbf{v}_B = 2\mathbf{j} + 20\mathbf{j}$

(a) ${}_A\mathbf{v}_B = \mathbf{v}_A - \mathbf{v}_B = 5\mathbf{i} - 30\mathbf{j}$km/h

(b) ${}_{B}\mathbf{v}_{A} = \mathbf{v}_{B} - \mathbf{v}_{A} = -5\mathbf{i} + 30\mathbf{j}\text{km/h}$

12. (a) ${}_{A}\mathbf{v}_{B} = \mathbf{v}_{A} + (-\mathbf{v}_{B})$



$$\begin{aligned}\alpha &= 360 - 90 - 120\\ &= 150°\end{aligned}$$

By the cosine rule,

$$\begin{aligned}|{}_{A}\mathbf{v}_{B}|^2 &= 12^2 + 15^2 - 2 \times 12 \times 15 \cos 150°\\ |{}_{A}\mathbf{v}_{B}| &= 26.09\text{km/h}\end{aligned}$$

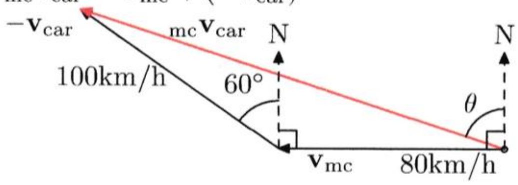
By the sine rule (unambiguous here),

$$\begin{aligned}\frac{\sin(\theta - 90)}{15} &= \frac{\sin 150°}{26.09}\\ \theta - 90 &= \sin^{-1}\frac{15\sin 150°}{26.09}\\ &= 16.71°\\ \theta &= 106.71°\end{aligned}$$

${}_{A}\mathbf{v}_{B} = 26.1\text{km/h}$ on a bearing of 253°.

(b) ${}_{B}\mathbf{v}_{A} = \mathbf{v}_{B} + (-\mathbf{v}_{A}) = -{}_{A}\mathbf{v}_{B}$
(Same diagram with all the arrows reversed.)
${}_{B}\mathbf{v}_{A} = 26.1\text{km/h}$ on a bearing of 073°.

13. ${}_{mc}\mathbf{v}_{car} = \mathbf{v}_{mc} + (-\mathbf{v}_{car})$



By the cosine rule,

$$\begin{aligned}|{}_{mc}\mathbf{v}_{car}|^2 &= 100^2 + 80^2 - 2 \times 100 \times 80 \cos 150°\\ |{}_{mc}\mathbf{v}_{car}| &= 173.94\text{km/h}\end{aligned}$$

By the sine rule (unambiguous here),

$$\begin{aligned}\frac{\sin(90 - \theta)}{100} &= \frac{\sin 150°}{173.94}\\ 90 - \theta &= \sin^{-1}\frac{100\sin 150°}{173.94}\\ &= 16.91°\\ \theta &= 73.29°\end{aligned}$$

${}_{mc}\mathbf{v}_{car} = 174\text{km/h}$ on a bearing of 287°.

14. ${}_{A}\mathbf{v}_{B} = \mathbf{v}_{A} - \mathbf{v}_{B}$
$\mathbf{v}_{B} = \mathbf{v}_{A} - {}_{A}\mathbf{v}_{B} = \mathbf{i} - 12\mathbf{j}$

15. ${}_{B}\mathbf{v}_{A} = \mathbf{v}_{B} - \mathbf{v}_{A}$
$\mathbf{v}_{B} = \mathbf{v}_{A} + {}_{B}\mathbf{v}_{A} = -2\mathbf{i} + 6\mathbf{j}$

16. LHS:

$$\begin{aligned}{}_{A}\mathbf{v}_{B} + {}_{B}\mathbf{v}_{C} &= (\mathbf{v}_{A} - \mathbf{v}_{B}) + (\mathbf{v}_{B} - \mathbf{v}_{C})\\ &= \mathbf{v}_{A} - \mathbf{v}_{B} + \mathbf{v}_{B} - \mathbf{v}_{C}\\ &= \mathbf{v}_{A} - \mathbf{v}_{C}\\ &= {}_{A}\mathbf{r}_{C}\\ &= \text{RHS. Q.E.D.}\end{aligned}$$

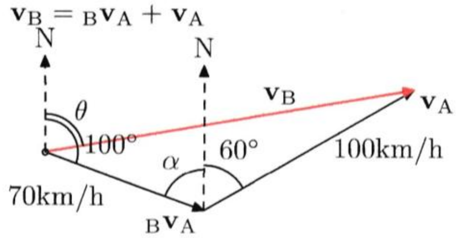
17. ${}_{A}\mathbf{v}_{B} = \mathbf{v}_{A} - \mathbf{v}_{B}$

$$\begin{aligned}\mathbf{v}_{B} &= \mathbf{v}_{A} - {}_{A}\mathbf{v}_{B}\\ &= 20 - 30\\ &= -10\text{km/h due North}\\ &= 10\text{km/h due South}\end{aligned}$$

18. ${}_{A}\mathbf{v}_{B} = \mathbf{v}_{A} - \mathbf{v}_{B}$

$$\begin{aligned}\mathbf{v}_{B} &= \mathbf{v}_{A} - {}_{A}\mathbf{v}_{B}\\ &= 80 - 60\\ &= 20\text{km/h due South}\end{aligned}$$

19. $\mathbf{v}_{B} = {}_{B}\mathbf{v}_{A} + \mathbf{v}_{A}$



$$\begin{aligned}\alpha &= 180 - 100 \text{ (cointerior angles)}\\ &= 80°\\ \alpha + 60 &= 140°\end{aligned}$$

By the cosine rule,

$$\begin{aligned}|\mathbf{v}_{B}|^2 &= 70^2 + 100^2 - 2 \times 70 \times 100 \cos 140°\\ |{}_{A}\mathbf{v}_{B}| &= 160.08\text{km/h}\end{aligned}$$

By the sine rule (unambiguous here),

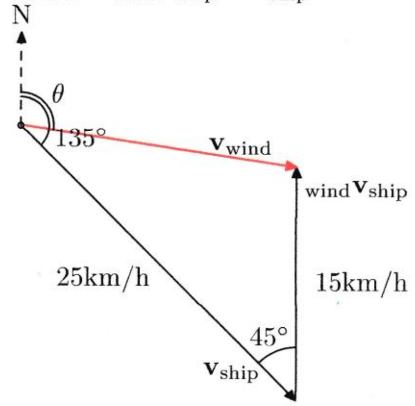
$$\begin{aligned}\frac{\sin(100° - \theta)}{100} &= \frac{\sin 140°}{160.08}\\ 100° - \theta &= \sin^{-1}\frac{100\sin 140°}{160.08}\\ &= 23.68°\\ \theta &= 100 - 23.68\\ &= 76.32°\end{aligned}$$

$\mathbf{v}_{B} = 160\text{km/h}$ on a bearing of 076°.

20. $\mathbf{v}_{wind} = {}_{wind}\mathbf{v}_{ship} + \mathbf{v}_{ship} = (13\mathbf{i} + \mathbf{j})\text{km/h}$

21. $\mathbf{v}_{wind} = {}_{wind}\mathbf{v}_{walker} + \mathbf{v}_{walker} = (4\mathbf{i} - 2\mathbf{j})\text{km/h}$

22. $\mathbf{v}_{\text{wind}} = {}_{\text{wind}}\mathbf{v}_{\text{ship}} + \mathbf{v}_{\text{ship}}$



By the cosine rule,

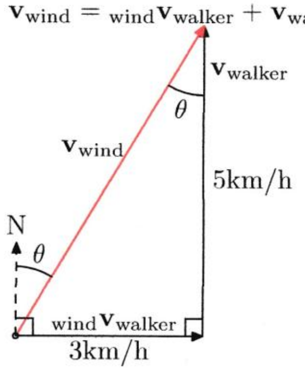
$$|\mathbf{v}_{\text{wind}}|^2 = 25^2 + 15^2 - 2 \times 25 \times 15 \cos 45^\circ$$
$$|\mathbf{v}_{\text{wind}}| = 17.88\text{km/h}$$

By the sine rule (unambiguous here),

$$\frac{\sin(135^\circ - \theta)}{15} = \frac{\sin 45^\circ}{17.88}$$
$$135^\circ - \theta = \sin^{-1} \frac{15 \sin 45^\circ}{17.88}$$
$$= 36.39^\circ$$
$$\theta = 135 - 36.39$$
$$= 98.61^\circ$$

$\mathbf{v}_{\text{wind}} = 17.9$km/h on a bearing of 099°. However, it is usual to give the direction a wind blows *from* so:
$\mathbf{v}_{\text{wind}} = 17.9$km/h blowing from 279°.

23. $\mathbf{v}_{\text{wind}} = {}_{\text{wind}}\mathbf{v}_{\text{walker}} + \mathbf{v}_{\text{walker}}$



By Pythagoras' Theorem,

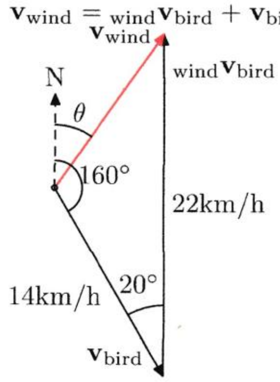
$$|\mathbf{v}_{\text{wind}}|^2 = 5^2 + 3^2$$
$$|\mathbf{v}_{\text{wind}}| = \sqrt{34}\text{km/h}$$
$$\approx 5.83\text{km/h}$$

$$\tan\theta = \frac{3}{5}$$
$$\theta = 30.96^\circ$$

$\mathbf{v}_{\text{wind}} = 5.8$km/h on a bearing of 031°. However, it is usual to give the direction a wind blows *from* so:
$\mathbf{v}_{\text{wind}} = 5.8$km/h blowing from 211°.

24.
- B appears stationary, so $\mathbf{v}_B = \mathbf{v}_A$ and B is travelling due north at 10km/h.
- C appears to be moving south at 3km/h. This means that C is actually moving north 3km/h slower than A. C is travelling due north at 7km/h.
- D appears to be moving north at 5km/h. It is actually travelling north 5km/h faster than A. D is travelling due north at 15km/h.

25. $\mathbf{v}_{\text{wind}} = {}_{\text{wind}}\mathbf{v}_{\text{bird}} + \mathbf{v}_{\text{bird}}$



By the cosine rule,

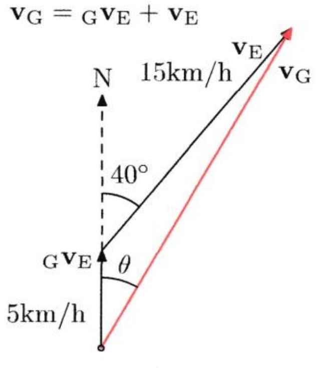
$$|\mathbf{v}_{\text{wind}}|^2 = 14^2 + 22^2 - 2 \times 14 \times 22 \cos 20^\circ$$
$$|\mathbf{v}_{\text{wind}}| = 10.06\text{km/h}$$

Without drawing a scale diagram, we don't know whether the angle $160^\circ - \theta$ is acute or obtuse, so the sine rule is ambiguous here. We'll use the cosine rule instead.

$$160^\circ - \theta = \cos^{-1} \frac{14^2 + 10.06^2 - 22^2}{2 \times 14 \times 10.06}$$
$$= 131.57^\circ$$
$$\theta = 28.43^\circ$$

$\mathbf{v}_{\text{wind}} = 10.1$km/h from 208°.

26.
- Ship F appears to be moving at the same speed as ship E but in the opposite direction ($220^\circ - 20^\circ = 180^\circ$). This implies that ship F is actually stationary.
- $\mathbf{v}_G = {}_G\mathbf{v}_E + \mathbf{v}_E$

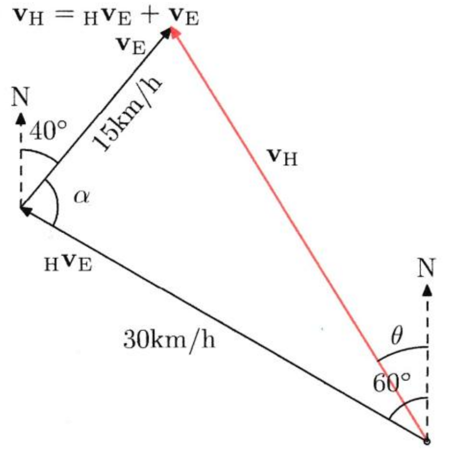


$$|\mathbf{v}_G|^2 = 5^2 + 15^2 - 2 \times 5 \times 15 \cos 140^\circ$$
$$|\mathbf{v}_G| = 19.10\text{km/h}$$

$$\frac{\sin(\theta)}{15} = \frac{\sin 140^\circ}{19.10}$$
$$\theta = \sin^{-1} \frac{15 \sin 140^\circ}{19.10}$$
$$= 30.31^\circ$$

$\mathbf{v}_G = 19.1$km/h on a bearing of 030°.

- $\mathbf{v}_H = {}_H\mathbf{v}_E + \mathbf{v}_E$



$$40 + \alpha = 180 - 60 \text{ (cointerior angles)}$$
$$\alpha = 80^\circ$$

$$|\mathbf{v}_H|^2 = 30^2 + 15^2 - 2 \times 30 \times 15 \cos 80^\circ$$
$$|\mathbf{v}_H| = 31.12\text{km/h}$$

$$\frac{\sin(60^\circ - \theta)}{15} = \frac{\sin 80^\circ}{31.12}$$
$$60 - \theta = \sin^{-1} \frac{15 \sin 80^\circ}{31.12}$$
$$= 28.33^\circ$$
$$\theta = 60 - 28.33$$
$$= 31.67^\circ$$

$\mathbf{v}_H = 31.1$km/h on a bearing of 328°.

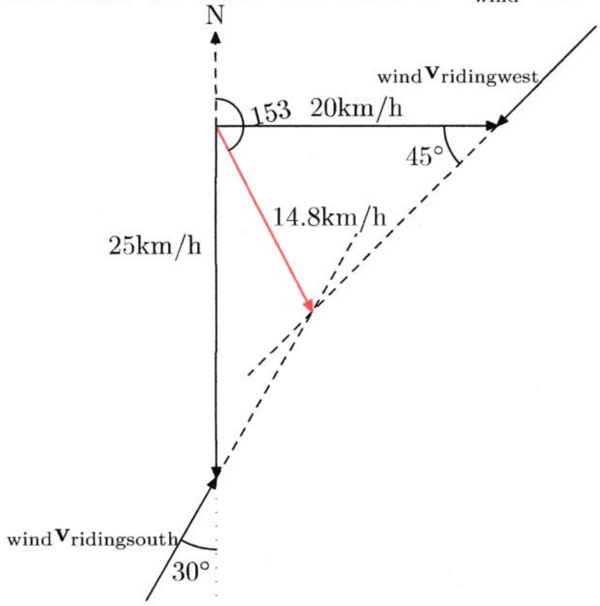
27. ${}_A\mathbf{v}_B = 7\mathbf{i} - 10\mathbf{j}$

$${}_A\mathbf{v}_C = 13\mathbf{i} - 2\mathbf{j}$$
$${}_B\mathbf{v}_C = \mathbf{v}_B - \mathbf{v}_C$$
$$= \mathbf{v}_B - \mathbf{v}_C + \mathbf{v}_A - \mathbf{v}_A$$
$$= \mathbf{v}_B - \mathbf{v}_A + \mathbf{v}_A - \mathbf{v}_C$$
$$= {}_B\mathbf{v}_A + {}_A\mathbf{v}_C \text{ (we might have started here)}$$
$$= -(7\mathbf{i} - 10\mathbf{j}) + (13\mathbf{i} - 2\mathbf{j})$$
$$= 6\mathbf{i} + 8\mathbf{j}$$

28. On the same diagram, draw the vectors representing the speed and direction of the cyclist travelling east and travelling south. From the head of each of these vectors draw the line representing the apparent direction of the wind. Since

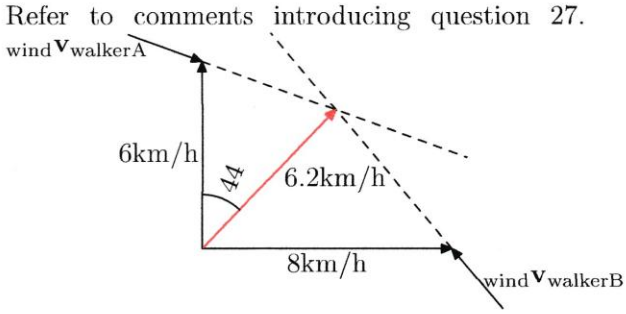
$$\mathbf{v}_{\text{wind}} = \mathbf{v}_{\text{cyclist}} + {}_{\text{wind}}\mathbf{v}_{\text{cyclist}}$$

with its foot at the origin, the head of $\mathbf{v}_{\text{wind}}$ must lie along this line. Thus, the point where these two extended lines intersect must be where the head of $\mathbf{v}_{\text{wind}}$ lies.



The wind velocity is 14.8km/h from 333°.

29. Refer to comments introducing question 27.



The wind velocity is 6.2km/h from 224°.